Review

- Atomic tableaux
- CST and properties
Outline

- Deduction from premises
- Syntax and semantics
- Soundness theorem
- Completeness theorem
Consequence

Definition

Let $\Sigma$ be a (possibly infinite) set of propositions. We say that $\sigma$ is a consequence of $\Sigma$ (and write as $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

$$(\forall \tau \in \Sigma \mathcal{V}(\tau) = T) \Rightarrow \mathcal{V}(\sigma) = T.$$
Consequence

Example
Example

Let $\Sigma = \{A, \neg A \lor B\}$, we have $\Sigma \models B$. 
Consequence

Example

1. Let $\Sigma = \{A, \neg A \lor B\}$, we have $\Sigma \models B$.
2. Let $\Sigma = \{A, A \rightarrow B\}$, we have $\Sigma \models B$. 
Let $\Sigma = \{A, \neg A \lor B\}$, we have $\Sigma \models B$.

Let $\Sigma = \{A, A \rightarrow B\}$, we have $\Sigma \models B$.

Let $\Sigma = \{\neg A\}$, we have $\Sigma \models (A \rightarrow B)$.
Deductions from Premises

How to construct CST from premises?

Definition (Tableaux from premises)

Let be (possibly infinite) set of propositions. We define the finite tableaux with premises from by induction:

1. Every atomic tableau is a finite tableau from

2. If is a finite tableau from and , then the tableau formed by putting at the end of every noncontradictory path not containing it is also a finite tableau from.
How to construct CST from premises?

Definition (Tableaux from premises)

Let $\Sigma$ be (possibly infinite) set of propositions. We define the finite tableaux with premises from $\Sigma$ by induction:

1. Every atomic tableau is a finite tableau from $\Sigma$
Deductions from Premises

How to construct CST from premises?

Definition (Tableaux from premises)

Let $\Sigma$ be (possibly infinite) set of propositions. We define the finite tableaux with premises from $\Sigma$ by induction:

1. Every atomic tableau is a finite tableau from $\Sigma$.
2. If $\tau$ is a finite tableau from $\Sigma$ and $\alpha \in \Sigma$, then the tableau formed by putting $T\alpha$ at the end of every noncontradictory path not containing it is also a finite tableau from $\Sigma$. 
If $\tau$ is a finite tableau from $\Sigma$, $P$ a path in $\tau$, $E$ an entry of $\tau$ occurring on $P$ and $\tau'$ is obtained from $\tau$ by adjoining the unique atomic tableau with root entry $E$ to the end of the path $P$, then $\tau'$ is also a finite tableau from $\Sigma$.

If $\tau_0, \ldots, \tau_n, \ldots$ is a (finite or infinite) sequence of finite tableaux from $\Sigma$ such that, for each $n \geq 0$, $\tau_{n+1}$ is constructed from $\tau_n$ by an application of (2) and (3), then $\tau = \bigcup \tau_n$ is a tableau from $\Sigma$. 
Tableau proof

Definition

A tableau proof of a proposition $\alpha$ from $\Sigma$ is a tableau from $\Sigma$ with root entry $F\alpha$ that is contradictory, that is, one in which every path is contradictory. If there is such a proof we say that $\alpha$ is provable from $\Sigma$ and write it as $\Sigma \vdash \alpha$. 
Theorem

Every CST from a set of premises is finished.
Example

Give you two Chinese characters “更衣”, what’s it mean?

It means change clothes in modern Chinese.
It means go to washroom in ancient Chinese.

Example

Give an acronym “IP”, what’s it mean?

Internet Protocol in network.
Integer Programming in operation research.
Interactive proof in complexity.
Example

Give you two Chinese characters “更衣”, what’s it mean?

- It means *change clothes* in modern Chinese.

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Example

Give you two Chinese characters “更衣”, what’s it mean?

- It means **change clothes** in modern Chinese.
- It means **go to washroom** in ancient Chinese.

Example

Give an acronym ”IP”, what’s it mean?
Example

Give you two Chinese characters “更衣”, what’s it mean?

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Example

Give you the following programming segments:
Example

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1. in C, printf("Hello World!");
Example

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1. in C, printf("Hello World!");
2. in Java, system.print("Hello World!");
Example

Give you the following programming segments:

1. in C, `printf("Hello World!");`
2. in Java, `system.print("Hello World!");`
3. in C++, `cout << "Hello World!";`
Syntax & Semantics

Example

Give you the following programming segments:

1. in C, printf("Hello World!");
2. in Java, system.print("Hello World!");
3. in C++, cout<<"Hello World!";

All of them just output "Hello World!" on the screen.
What’s syntax?
Syntax & Semantics in PL

- What’s syntax?
- What’s semantic?
What’s syntax?
What’s semantic?
What’s relationship between them?
Soundness

Example

Consider Pierce Law

\[ ((A \rightarrow B) \rightarrow A) \rightarrow A. \]
Example
Consider Pierce Law

\[ ((A \rightarrow B) \rightarrow A) \rightarrow A. \]

- Give its tableau proof.
Soundness

Example
Consider Pierce Law

\[(A \rightarrow B) \rightarrow A) \rightarrow A.\]

- Give its tableau proof.
- Give its truth table.
Example

Given proposition \(((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow C)\), there is a truth valuation which make it false.
Example

Given proposition \(((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow C)\), there is a truth valuation which make it false.

Consider the tableau with the root as

\[ F ((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow C) \]
Lemma

If V is a valuation that agrees with the root entry of a given tableau $\tau$ given as $\bigcup \tau_n$, then $\tau$ has a path $P$ every entry of which agrees with $V$. 
Theorem (Soundness)

If $\alpha$ is tableau provable, then $\alpha$ is valid, i.e. $\vdash \alpha \Rightarrow \models \alpha$. 
Lemma

If a valuation $\mathcal{V}$ makes every $\alpha \in \Sigma$ true and agrees with the root of a tableau $\tau$ from $\Sigma$, then there is a path in $\tau$ every entry of which agrees with $\mathcal{V}$. 
Soundness of deductions from premises

**Theorem**

*If there is a tableau proof of $\alpha$ from a set of premises $\Sigma$, then $\alpha$ is a consequence of $\Sigma$, i.e. $\Sigma \vdash \alpha \Rightarrow \Sigma \models \alpha$.***
Example

Given proposition \(((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow C)\), there is a truth valuation which make it false. Observe the non-contradictory path of the tableau with the root as \(F ((A \rightarrow B) \rightarrow (A \rightarrow C)) \rightarrow (B \rightarrow C)\)
Lemma

Let $P$ be a noncontradictory path of a finished tableau $\tau$. Define a truth assignment $A$ on all propositional letters $A$ as follows:

1. $A(A) = T$ if $TA$ is an entry on $P$.
2. $A(A) = F$ otherwise.

If $V$ is the unique valuation extending the truth assignment $A$, then $V$ agrees with all entries of $P$. 
**Completeness**

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Discrete Mathematics

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Completeness

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1. $A(A) = T$ if $TA$ is an entry on $P$.
2. $A(A) = F$ otherwise.

If $\nu$ is the unique valuation extending the truth assignment $A$, then $\nu$ agrees with all entries of $P$. 
Theorem (Completeness)

If $\alpha$ is valid, then $\alpha$ is tableau provable, i.e. $\models \alpha \implies \vdash \alpha$. In fact, any finished tableau with root entry $F\alpha$ is a proof of $\alpha$ and so, in particular, the complete systematic tableaux with root $F\alpha$ is such a proof.
Completeness of deduction from premises

Lemma

Let $P$ be a noncontradictory path in a finished tableau $\tau$ from $\Sigma$. Define a valuation $\mathcal{V}$ as the last section, then it agrees with all entries on $P$ and so in particular makes every proposition $\beta \in \Sigma$ true.
Completeness of deduction from premises

**Theorem**

If $\alpha$ is consequence of a set $\Sigma$ of premises, then there is a tableau deduction of $\alpha$ from $\Sigma$, i.e., $\Sigma \vDash \alpha \Rightarrow \Sigma \vdash \alpha$. 
Hilbert Proof System

**Definition**

The axioms of Hilbert system are all propositions of the following forms:

1. \((\alpha \rightarrow (\beta \rightarrow \alpha))\)
2. \(((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))\)
3. \((\neg \beta \rightarrow \neg \alpha) \rightarrow ((\neg \beta \rightarrow \alpha) \rightarrow \beta)\)
Definition (Modus Ponens)

From $\alpha$ and $\alpha \rightarrow \beta$, we can infer $\beta$. This rule is written as follows:

\[
\begin{array}{c}
\alpha \\
\alpha \rightarrow \beta \\
\hline
\beta \\
\end{array}
\]
Definition

Let \( \Sigma \) be a set of propositions.

1. A \textit{proof from} \( \Sigma \) is a finite sequence \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that for each \( i \leq n \) either:
   1. \( \alpha_i \) is a member of \( \Sigma \).
   2. \( \alpha_i \) is an axiom;
   or
   3. \( \alpha_i \) can be inferred from some of previous \( \alpha_j \) by an application of a rule of inference.

2. \( \alpha \) is \textit{provable from} \( \Sigma \), \( \Sigma \vdash_{H} \alpha \), if there is a proof \( \alpha_1, \alpha_2, \ldots, \alpha_n \) from \( \Sigma \) where \( \alpha_n = \alpha \).

3. A \textit{proof} of \( \alpha \) is simply a proof from the empty set \( 0 \); \( \alpha \) is \textit{provable} if it is provable from \( 0 \).
Next Class

- Compactness
- Application