Review of Lattice

- Ideal
- Special Lattice
- Boolean Algebra
Examples of Proof

- Zeno’s paradox
Examples of Proof

- Zeno’s paradox
- Zhuang Zi’s paradox
Examples of Proof

- Zeno’s paradox
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- Gongsun Long’s “a white horse is not a horse”
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- ...
Examples of Proof

- Zeno’s paradox
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How can you persuade yourself and the others?
Examples of Proof

Axiom

The axiom of group theory can be formulated as follows:

\begin{align*}
(G1) & \quad (x \circ y) \circ z = x \circ (y \circ z) \\
(G2) & \quad x \circ e = x \\
(G3) & \quad \text{For every } x \text{ there is a } y \text{ such that } x \circ y = e \text{ (right inverse)}
\end{align*}

Theorem

For every \( x \) there is a \( y \) such that \( y \circ x = e \). (left inverse)
Axiom

The axiom of group theory can be formulated as follows:

\((G1)\)  For all \(x, y, z\): \((x \circ y) \circ z = x \circ (y \circ z)\).
Examples of Proof

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(G1) For all $x, y, z$: $(x \circ y) \circ z = x \circ (y \circ z)$.

(G2) For all $x$: $x \circ e = x$. 

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Examples of Proof

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(G1) For all $x, y, z$: $(x \circ y) \circ z = x \circ (y \circ z)$.

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For every \(x\) there is a \(y\) such that \(y \circ x = e\). (left inverse)
What is Logic

- Premise
What is Logic

- Premise
- Argument
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- Premise
- Argument
- Conclusion
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- Premise
- Argument
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- Follow
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- Proof
Aristotle (384-322 B.C.): theory of syllogistic
History of Mathematical Logic

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- Robinson (1930-); Beth and Smullyan
- Leibniz (1646-1716) and Hilbert (1862-1943)
Introduction to Mathematical Logic

- First order logic
First order logic
  • Propositional Logic
Introduction to Mathematical Logic

- First order logic
  - Propositional Logic
  - Predicate Logic
Introduction to Mathematical Logic

- First order logic
  - Propositional Logic
  - Predicate Logic

- High order logic
Introduction to Mathematical Logic

- First order logic
  - Propositional Logic
  - Predicate Logic

- High order logic
- Other type of logic
Introduction to Mathematical Logic

- First order logic
  - Propositional Logic
  - Predicate Logic
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  - Propositional Logic
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  - Intuitionistic logic
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  - Propositional Logic
  - Predicate Logic
- High order logic
- Other type of logic
  - Modal logic
  - Intuitionistic logic
  - Temporal logic
Introduction to Mathematical Logic

- Proof system
Proof system
- Axiom
Proof system
  - Axiom
  - Tableaux
Introduction to Mathematical Logic

- Proof system
  - Axiom
  - Tableaux
  - Resolution
Introduction to Mathematical Logic

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  - Tableaux
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- Two Components
Introduction to Mathematical Logic

- Proof system
  - Axiom
  - Tableaux
  - Resolution
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  - Syntax
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  - Syntax
  - Semantics
Introduction to Mathematical Logic

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- Algorithmic approach
Programming language
Application

1. Programming language
2. Digital circuit
Application

1. Programming language
2. Digital circuit
3. Database
Application

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3. Database
4. Program verification
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5. Computational theory
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5. Computational theory
6. Artificial intelligence
Reference Book

1. Hao Wang, Reflections on Kurt Gödel
2. Hao Wang, A Logical Journey: From Gödel to Philosophy
3. Huth&Ryan, Logic in Computer Science 2ed
Definition (Partial order)

A partial order is a set $S$ with a binary relation $<$ on $S$, which is transitive and irreflexive.
Order

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A partial order is a set $S$ with a binary relation $<$ on $S$, which is *transitive* and *irreflexive*.

Definition (Linear order)

A partial order $<$ is a linear order, if it satisfies the *trichotomy* law: $x < y$ or $x = y$ or $y < x$. 
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A partial order is a set $S$ with a binary relation $<$ on $S$, which is transitive and irreflexive.

Definition (Linear order)
A partial order $<$ is a linear order, if it satisfies the trichotomy law: $x < y$ or $x = y$ or $y < x$.

Definition (Well ordering)
A linear order is well ordered if every nonempty set $A$ of $S$ has a least element.
Definition (Countable)
A set $A$ is *countable* if there is a one-to-one mapping from $A$ to $\mathbb{N}$.

Definition (Finite)
A set $A$ is *finite* if there is a one-to-one mapping from $A$ to $\{0; 1; \ldots; n\}$ for some $n \in \mathbb{N}$.

If $A$ is not countable, it is *uncountable*.

If $A$ is not finite, it is *infinite*.
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Definition
1. If $A$ is not countable, it is *uncountable*.
2. If $A$ is not finite, it is *infinite*.
Theorem

Let $A$ be a countable set. The set of all finite sequence of elements in $A$ is also countable.
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Let $A$ be a countable set. The set of all finite sequence of elements in $A$ is also countable.

Proof.

We can formalize it as

$$S = \bigcup_{n \in \mathbb{N}} A^n = A^1 \cup A^2 \cup \ldots \cup A^n \cup \ldots.$$ 

Construct a mapping from $A^n$ to $\mathbb{N}$. 

Definition (Tree)

A tree is a set $T$ (whose elements are called nodes) partially ordered by $<_T$, with a unique least element called the root, in which the predecessors of every node are well ordered by $<_T$. 
Definition (Tree)

A 

A path on a tree is a maximal linearly ordered subset of .
The levels of a tree $T$ are defined by induction.

The 0th level of $T$ consists precisely of the root of $T$.

The $k+1$th level of $T$ consists of the immediate successors of the nodes on the $k$th level of $T$. 
The levels of a tree $T$ are defined by induction.
Definition (Properties of tree)

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2. The $0^{th}$ level of $T$ consists precisely of the root of $T$.
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The depth of a tree $T$ is the maximum $n$ such that there is a node of level $n$ in $T$. If there are nodes of the level $n$ for every natural number $n$, we say the depth of $T$ is infinite.
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Definition (Properties of tree)

1. The *depth* of a tree $T$ is the maximum $n$ such that there is a node of level $n$ in $T$.

2. If there are nodes of the level $n$ for every natural number $n$, we say the depth of $T$ is infinite or $\omega$. 
Definition (Properties of tree)

1. If each node has at most $n$ immediate successors, the tree is $n$-ary or $n$-branching.
2. If each node has finitely many immediate successors, we say that the tree is finitely branching.
3. A node with no successors is called a leaf or a terminal node.
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Theorem (König’s lemma)

If a finitely branching tree $T$ is infinite, it has an infinite path.
Tree

Theorem (König’s lemma)

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Proof.
Theorem (König’s lemma)

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Proof.

1. If there is no infinite path, the tree would be finite.
Theorem (König’s lemma)

If a finitely branching tree $T$ is infinite, it has an infinite path.

Proof.

1. If there is no infinite path, the tree would be finite.
2. Split the successors of the node into two parts. One with infinite successors and the other with finite successors.
A labeled tree $T$ is a tree $T$ with a function (the labeling function) that associates some objects with every node. This object is called the label of the node.
Definition (Segment)

1. is an initial segment of if or .

2. is a proper initial segment of if .

Definition (Lexicographic ordering)

For two sequences and we say that if or if (n) where n is the first entry at which the sequences differ.
Definition (Segment)

1. $\sigma$ is an *initial segment* of $\tau$ if $\sigma \subseteq \tau$ or $\sigma = \tau$. 
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1. $\sigma$ is an initial segment of $\tau$ if $\sigma \subseteq \tau$ or $\sigma = \tau$.
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2. $\sigma$ is an proper initial segment of $\tau$ if $\sigma \subset \tau$.

Definition (Lexicographic ordering)

For two sequences $\sigma$ and $\tau$ we say that $\sigma <_L \tau$ if $\sigma \subseteq \tau$ or if $\sigma(n)$, the $n^{th}$ entry in $\sigma$, is less than $\tau(n)$ where $n$ is the first entry at which the sequences differ.
One way to define a linear order based on given tree.

Definition (left to right ordering)

Given two nodes $x$ and $y$, 

1. If $x < T y$, we say that $x < L y$.
2. If $x$ and $y$ are incomparable in the tree ordering, find the largest predecessors of $x$ and $y$ on the same level, say $x'$ and $y'$.
   1. If $x' = y'$, $x \leq L y$ if and only if $x$ is left to $y$ relative to $x'$.
   2. Otherwise $x < L y$ if and only if $x' < L y'$. 

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Tree

One way to define a linear order based on given tree.

Definition (left to right ordering)

Given two nodes $x$ and $y$,

1. If $x \prec_T y$, we say that $x \prec_L y$.
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   1. If $x'$ equals $y'$, $x \leq y$ if and only if $x$ is left to $y$ relative to $x'$.
One way to define a linear order based on given tree.

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   1. If $x'$ equals $y'$, $x \leq y$ if and only if $x$ is left to $y$ relative to $x'$.
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Next Class

- Language of proposition logic
- Formation tree
- Truth table
- Connectives