Outline

- Introduction
- Review of partial order set
- Review of abstract algebra
- Lattice and Sublattice
Introduction

Why do we learn lattice?

- A special algebraic structure
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- Logic (Model theory)
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- Logic (Model theory)
- Compiler (Program analysis)
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- ...

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Discrete Mathematics
February 27, 2017
Introduction

- Intensively explored area
  1. By 1960s, 1,500 papers and books
  2. By 1970s, 2,700 papers and books
  3. By 1980s, 3,200 papers and books
  4. By 1990s, 3,600 papers and books

- History
  1. By 1850, George Boole’s attempt to formalize proposition logic.
  2. At the end of 19th century, Charles S. Pierce and Ernst Schröder
  4. Until mid-1930’s, Garrett Birkhoff developed general theory on lattice.
Partial order set (Poset)

**Definition**

Given a set $A$ and a relation $R$ on it, $< A, R >$ is called a partially ordered set (poset in brief) if $R$ is *reflexive*, *antisymmetric* and *transitive*. 
Definition

Given a poset $< A, \leq >$, we can define:

1. $a$ is maximal if there does not exist $b \in A$ such that $a \leq b$.
2. $a$ is minimal if there does not exist $b \in A$ such that $b \leq a$.
3. $a$ is greatest if for every $b \in A$, we have $b \leq a$.
4. $a$ is least if for every $b \in A$, we have $a \leq b$. 
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Definition

Given a poset $< A, \leq >$ and a set $S \subseteq A$. 

- $u \in A$ is an upper bound of $S$ if $s \leq u$ for every $s \in S$.
- $l \in A$ is a lower bound of $S$ if $l \leq s$ for every $s \in S$. 
Definition

Given a poset \( < A, \leq > \) and a set \( S \subseteq A \).

\( u \in A \) is a \textit{upper bound} of \( S \) if \( s \leq u \) for every \( s \in S \).
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Given a poset \( < A, \leq > \) and a set \( S \subseteq A \).

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Given a poset $< A, \leq >$ and a set $S \subseteq A$. 

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Definition

Given a poset \( < A, \leq > \) and a set \( S \subseteq A \).

1. \( u \) is a \textit{least upper bound} of \( S \), \((LUB(S))\), if \( u \) is the upper bound of \( S \) and \( u \leq u' \) for any other upper bound \( u' \) of \( S \).
Definition

Given a poset \( < A, \leq > \) and a set \( S \subseteq A \).

1. \( u \) is a least upper bound of \( S \), \((LUB(S))\), if \( u \) is the upper bound of \( S \) and \( u \leq u' \) for any other upper bound \( u' \) of \( S \).

2. \( l \) is a greatest lower bound of \( S \), \((GLB(S))\), if \( l \) is the upper bound of \( S \) and \( l' \leq l \) for any other lower bound \( l' \) of \( S \).
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Given a poset \( < A, \leq > \) and a set \( S \subseteq A \).

1. \( u \) is a least upper bound of \( S \), \( (LUB(S)) \), if \( u \) is the upper bound of \( S \) and \( u \leq u' \) for any other upper bound \( u' \) of \( S \).

2. \( l \) is a greatest lower bound of \( S \), \( (GLB(S)) \), if \( l \) is the upper bound of \( S \) and \( l' \leq l \) for any other lower bound \( l' \) of \( S \).

Theorem

A poset has at most one LUB or GLB.
**Definition**

A *lattice* (structure) is a poset $\langle A, \leq \rangle$ in which any two elements $a, b$ have a $\text{LUB}(a, b)$ and a $\text{GLB}(a, b)$.

From now on, we define $a \cup b = \text{LUB}(a, b)$ and $a \cap b = \text{GLB}(a, b)$ in brief. We also call them join and meet respectively.
Hasse diagram

Figure: Hasse diagram of $\langle \mathcal{P}(\{a, b\}), \subseteq \rangle$
1. Hasse diagram

Figure: Hasse diagram of $\langle \mathcal{P}(\{a, b\}), \subseteq \rangle$

2. Joint/meet table
1. Hasse diagram
2. Joint/meet table

\[
\begin{array}{c|cccc}
\cup & \emptyset & \{a\} & \{b\} & \{a, b\} \\
\emptyset & \emptyset & \{a\} & \{b\} & \{a, b\} \\
\{a\} & \{a\} & \{a\} & \{a, b\} & \{a, b\} \\
\{b\} & \{b\} & \{a, b\} & \{b\} & \{a, b\} \\
\{a, b\} & \{a, b\} & \{a, b\} & \{a, b\} & \{a, b\} \\
\end{array}
\]

Table: Joint table of \(\langle \mathcal{P}(\{a, b\}), \subseteq \rangle\)
1. Hasse diagram

2. Joint/meet table

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<th>{b}</th>
<th>{a, b}</th>
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<td>{b}</td>
<td>{b}</td>
</tr>
<tr>
<td>{a, b}</td>
<td>∅</td>
<td>{a}</td>
<td>{b}</td>
<td>{a, b}</td>
</tr>
</tbody>
</table>

Table: Meet table of \(\mathcal{P}(\{a, b\}), \subseteq\)
The Lattice has the following properties:

1. Commutative: \( a \lor b = b \lor a; a \land b = b \land a \).
2. Associative: \((a \lor b) \lor c = a \lor (b \lor c); (a \land b) \land c = a \land (b \land c)\).
3. Idempotent: \( a \lor a = a; a \land a = a \).
4. Absorption: \((a \land b) \lor a = a; (a \lor b) \land a = a\).
The Lattice has the following properties:

1. **Commutative**: \( a \cap b = b \cap a, a \cup b = b \cup a \).
The Lattice has the following properties:

1. **Commutative:** \(a \cap b = b \cap a, a \cup b = b \cup a\).
2. **Associative:**
   \[
   (a \cap b) \cap c = a \cap (b \cap c), (a \cup b) \cup c = a \cup (b \cup c).
   \]
The Lattice has the following properties:

1. **Commutative:** $a \cap b = b \cap a$, $a \cup b = b \cup a$.
2. **Associative:**
   
   $$(a \cap b) \cap c = a \cap (b \cap c), (a \cup b) \cup c = a \cup (b \cup c).$$
3. **Idempotent:** $a \cap a = a$, $a \cup a = a$. 

Property

A lattice could be divided into a join-semilattice and a meet-semilattice.
The Lattice has the following properties:

1. **Commutative:** \( a \cap b = b \cap a, \ a \cup b = b \cup a. \)
2. **Associative:**
   \[
   (a \cap b) \cap c = a \cap (b \cap c), \quad (a \cup b) \cup c = a \cup (b \cup c).
   \]
3. **Idempotent:** \( a \cap a = a, \ a \cup a = a. \)
4. **Absorption:** \( (a \cup b) \cap a = a, \ (a \cap b) \cup a = a. \)
The Lattice has the following properties:

1. **Commutative:** $a \cap b = b \cap a$, $a \cup b = b \cup a$.
2. **Associative:**
   
   $$(a \cap b) \cap c = a \cap (b \cap c), \quad (a \cup b) \cup c = a \cup (b \cup c).$$

3. **Idempotent:** $a \cap a = a$, $a \cup a = a$.
4. **Absorption:**
   
   $$(a \cup b) \cap a = a, \quad (a \cap b) \cup a = a.$$
A semilattice is an algebra $\mathcal{S} = (S, \star)$ satisfying, for all $x, y, z \in S$, 

1. $x \star x = x$,
2. $x \star y = y \star x$,
3. $x \star (y \star z) = (x \star y) \star z$. 
A semilattice is an algebra $\mathcal{S} = (S, \ast)$ satisfying, for all $x, y, z \in S$,

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Definition

A semilattice is an algebra $\mathcal{S} = (\mathcal{S}, \ast)$ satisfying, for all $x, y, z \in \mathcal{S}$,

1. $x \ast x = x$,
2. $x \ast y = y \ast x$,
3. $x \ast (y \ast z) = (x \ast y) \ast z$. 
Definition

Given a algebra $\mathcal{L} = (L, \cap, \cup)$, it is a lattice if it subjects to:

1. $L$ and $\cup$ are two semilattices.
2. $(a \cap b) \cup a = a$; $(a \cup b) \cap a = a$. 

Theorem

If $L$ is any set in which there are two operation defined as $\cap$ and $\cup$ satisfying the last four properties, then $L$ is a lattice.
Definition

Given a algebra \( \mathcal{L} = (L, \cap, \cup) \), it is a lattice if it subjects to:

1. \((L, \cup)\) and \((L, \cap)\) are two semilattices.
Lattice

Definition

Given a algebra $\mathcal{L} = (L, \cap, \cup)$, it is a lattice if it subjects to:

1. $(L, \cup)$ and $(L, \cap)$ are two semilattices.
2. $(a \cup b) \cap a = a$, $(a \cap b) \cup a = a$. 

Theorem

If $L$ is any set in which there are two operation defined as $\cap$ and $\cup$ satisfying the last four properties, then $L$ is a lattice.
Lattice

Definition
Given a algebra \( \mathcal{L} = (L, \cap, \cup) \), it is a lattice if it subjects to:

1. \((L, \cup)\) and \((L, \cap)\) are two semilattices.
2. \((a \cup b) \cap a = a\), \((a \cap b) \cup a = a\).

Theorem
If \( L \) is any set in which there are two operation defined as \( \cup \) and \( \cap \) satisfying the last four properties, then \( L \) is a lattice.
Definition
A subset \( S \) of a lattice \( L \) is called sublattice if it is closed under the operation \( \cup \) and \( \cap \).
Sublattice and extension

Definition
A subset $S$ of a lattice $L$ is called sublattice if it is closed under the operation $\cup$ and $\cap$.

Definition
If $S$ is a sublattice of $L$, $L$ is an extension of $S$. 
**Definition**

A subset $S$ of a lattice $L$ is called sublattice if it is closed under the operation $\cup$ and $\cap$.

**Definition**

If $S$ is a sublattice of $L$, $L$ is an extension of $S$.

**Definition**

The subset $S$ of the lattice $L$ is called *convex* if $a, b \in S, c \in L$, and $a \leq c \leq b$ imply that $c \in S$. 
Theorem

Given two lattice \( L \) and \( L' \), a bijection \( f : L \rightarrow L' \) from \( L \) to \( L' \) is an isomorphism if and only if \( a \leq b \) in \( L \) implies \( f(a) \leq f(b) \) in \( L' \).
Next Class

- Special lattices
- Boolean algebra